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# Analysis of the swimming of elastic slender bodies excited by an external force

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The essential difference, from the theoretical point of view, between an externally excited body and a fish is that the latter can apply lateral vibratory movements at any part of its surface, whereas in the 'artificial fish' lateral vibrations are applied only at the point where the external force acts on the body. A good example which illustrates how the artificial fish swims is the 'Pod'. The Pod is a medical device consisting of a small magnet attached to a plastic 'tail'. If the Pod is placed in a patient's blood vessel, and an alternating magnetic field is applied, the magnet oscillates angularly and the plastic tail causes it to swim. The purpose of the device is to deliver medicaments at any desired location in the circulatory system.

In this paper the theory of swimming of elastic slender bodies excited by an external force is presented. Special reference is made to the hydrodynamic forces acting on a swimming cylinder in viscous fluctuating flow. The results obtained are used in the analysis of the propulsion mechanism of the Pod.

## 1. Introduction

We consider a slender elastic body swimming with uniform velocity $U$ in an infinite viscous fluid. Figure 1 represents such a body of length $l$. Let us suppose that the motion is confined to the $x, z$ plane and let $h(x,t)$ denote the small lateral instantaneous displacement of the cross-section $A(x)$, where $0 \leqslant x \leqslant l$ and $t$ is the time. $F(x,t)$ is the time-dependent external force per unit length acting in the $z$ direction and $L(x,t)$ is the local hydrodynamic lift. From the theory of vibrations of elastic beams the equation of motion of the body can be put in the form

$$m(x)\frac{\partial^2 h}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left\{EI(x)\frac{\partial^2 h}{\partial x^2}\right\} = F(x,t)+L(x,t). \tag{1}$$

Here $m(x)$ denotes the mass per unit length of the material of the body, $E$ is Young's modulus and $I(x)$ is the moment of inertia of the cross-sectional area $A(x)$.

The thrust force $P$, which causes the body to swim, is calculated by integrating the product of the lateral forces and the instantaneous slope $\partial h/\partial x$ along the length of the body, i.e.

$$P = \int_0^l \{F(x,t)+L(x,t)\}\frac{\partial h}{\partial x}dx. \tag{2}$$

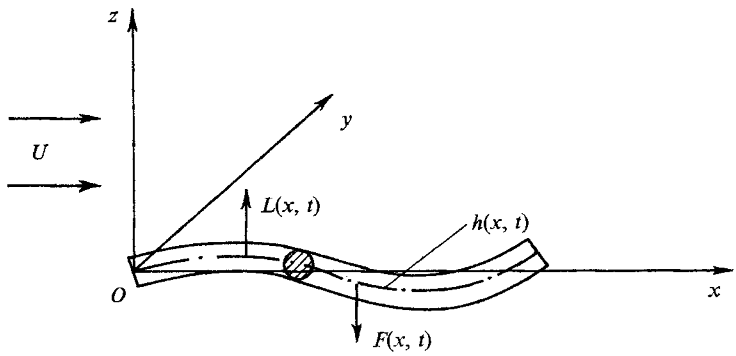


FIGURE 1. An elastic body of length $l$. The body is regarded as oscillating in the $z$ direction in a stream of velocity $U$ in the $Ox$ direction.

When the propulsion force is generated and the body starts moving, it appears that a drag force is being generated also. The drag $D$ is given by

$$D = \tfrac{1}{2}\rho C_D S U^2, \tag{3}$$

where $\rho$ is the density of the liquid, $C_D$ is the drag coefficient and $S$ is the reference area. When the drag balances the thrust we get $\overline{P} = D$, where $\overline{P}$ is the mean thrust averaged over a long time. From this equality we can determine the swimming velocity $U$.

Longman & Lavie (1966) analysed the swimming of the Pod by using the results obtained by Lighthill (1960$a$, $b$) for the hydrodynamic forces. From slender-body theory these forces were found to be

$$\left.\begin{aligned} L(x,t) &= -\rho\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left\{\tilde{A}(x)\left(\frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}\right)\right\},\\ \overline{P} &= \tfrac{1}{2}\rho\left\{\tilde{A}(x)\left[\overline{\left(\frac{\partial h}{\partial t}\right)^2}-U^2\overline{\left(\frac{\partial h}{\partial x}\right)^2}\right]\right\}_0^l, \end{aligned}\right\} \tag{4}$$

where the bars over the quantities denote their long-time averages and $\rho\tilde{A}(x)$ is the 'virtual mass' per unit length of a cylinder having cross-sectional area $A(x)$, for motions in the $z$ direction. Thus $\tilde{A}(x)$ is equal to $A(x)$ when the latter is circular, while for an ellipse with minor axis in the $z$ direction, $\tilde{A}(x)$ is the area of its circumscribing circle.

Longman & Lavie used equations (4) in the solution of (1) and calculated the thrust $\overline{P}$ accordingly. But since slender-body theory assumes the existence of a potential inviscid flow around the body and $L(x,t)$ does not include any term proportional to $\partial h/\partial t$ it follows that (1) describes a free undamped motion of the lateral displacement $h(x,t)$. Hence, when the frequency of the external force $F(x,t)$ coincides with the natural frequency of the body resonant conditions appear and the displacement $h(x,t)$ tends to infinity. In practice such conditions do not exist. This fact was also proved experimentally (see Lavie 1970$a$, $b$).

In order to improve the theoretical results Longman & Lavie suggested the inclusion in the expression for the instantaneous lift $L(x,t)$ of a term proportional to $\mu\,\partial h/\partial t$, where $\mu$ is the dynamic viscosity. Later on Lavie (1970$a$) developed a new model for the propulsion mechanism of a slender flexible cylinder in viscous flow. The new results for $L(x, t)$ and $\bar{P}$ agree well both with the necessity to damp out the persistence of free oscillations and with the experiments. When the viscosity is equal to zero the new forces coincide with the forces in (4). In the next section we shall describe briefly the paper Lavie (1970*a*) and in § 3 we shall apply the results to the theory of the swimming of the Pod. Also, a few more remarks are made about the swimming of slender bodies inside tubes and pipes.

## 2. The hydrodynamic forces acting on a flexible cylinder swimming in a viscous flow

Referring to figure 1, the flow field around the slender cylinder will be described by solving the Oseen equations. There are several reasons for using the Oseen equations in this particular case. First of all, we know that in order to calculate the lift of a two-dimensional cylinder vibrating perpendicularly to its longitudinal axis it is convenient to use the Stokes equation. Segel (1961) has remarked that the Stokes equations give better results than the boundary-layer equations when the lift is calculated. Stuart & Woodgate (1955) found that the theory is in good agreement with the experiments for a small lateral movement whose amplitude is 0·1 of the radius of the cylinder (for a more extensive discussion of the problem see Lavie (1970*a*)).

Here we treat the case when a uniform velocity $U$ is superimposed on the whole liquid–body system, and the use of the Oseen equations rather than the Stokes equations is essential. Kaplun (1957) and Kaplun & Lagerstrom (1957) have also pointed out that as the Reynolds number of the flow tends to zero the Oseen equations are uniformly valid. In time-dependent flow one must distinguish between two Reynolds numbers. One of them is related to the uniform velocity $U$ and the second one is related to the lateral velocity of the cylinder's cross-section. For the Pod the first Reynolds number ranges between $10^2$ and $10^4$ and the second Reynolds number is less than $10^2$. We shall assume that the analysis represented here is restricted to small lateral movements of the cross-section as found by Stuart & Woodgate (1955). Lavie (1970*a*) used the Oseen equations in three dimensions and applied certain assumptions about slender bodies.

For slender bodies the derivative of a certain hydrodynamic quantity in the $x$ direction is of the order $\epsilon$ in comparison with the derivatives in the $y$ and the $z$ directions, where $\epsilon = a/l$ and $a$ is the radius of the cylinder. In fact, if $u$, $v$ and $w$ are the fluid velocities in the $x$, $y$, and $z$ directions respectively, $\partial u/\partial x$ is of order $\epsilon^2$ relative to $\partial v/\partial y$ and $\partial w/\partial z$. To show this, let us assume for the moment that the flow is inviscid and has a potential $\phi$ such that $\partial u/\partial x \sim \epsilon^2 \partial^2\phi/\partial x^2$, while $\partial v/\partial y = \partial^2\phi/\partial y^2$ and $\partial w/\partial z = \partial^2\phi/\partial z^2$. Thus for slender bodies we can ignore the $x$ derivative in the equation of continuity and write

$$\frac{dv}{dy}+\frac{dw}{dz} \sim \frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz} = 0. \tag{5}$$

Equation (5) shows that a two-dimensional flow exists around any cross-section of the cylinder. The magnitude of this flow is determined by the distance $x$, by the time $t$ and, of course, by $y$ and $z$. Thus there exists a stream function $\psi(x, y, z, t)$ such that

$$v = \partial\psi/\partial z, \quad w = -\partial\psi/\partial y. \tag{6}$$

Referring to the Oseen equations and (5) and (6) we see that the equations for the velocity components $v$ and $w$ are independent of $u$. Therefore we can treat the following two equations independently of the components in the $x$ direction in the Oseen equations.

$$\left.\begin{aligned} \frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial y}-\nu\Delta_2 v &= 0,\\ \frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial z}-\nu\Delta_2 w &= 0,\\ \Delta_2 p &= 0,\end{aligned}\right\} \tag{7}$$

where $p$ is the pressure, $\nu$ is the kinematic viscosity and

$$\Delta_2 = \partial^2/\partial y^2 + \partial^2/\partial z^2. \tag{8}$$

Equations (7) are the equations of the flow around the cross-section at given $x$. One boundary condition related to these equations is that at infinity the velocity components $v$ and $w$ become zero and the pressure $p$ is equal to the static pressure $p_\infty$. Another boundary condition is that on the cylinder the velocity component $w$ should be equal to the lateral velocity of the cylinder's cross-section $\partial h/\partial t$, where $h(x, t)$ is the lateral displacement. However, the last condition is somewhat questionable. If, for instance, we had potential flow, and liquid slipping on the cylinder surface were possible, then the lateral velocity would be $\partial h/\partial t + U\,\partial h/\partial x$, where the term $U\,\partial h/\partial x$ accounts for the additional velocity due to the local instantaneous slope. In actual practice, if separation does not occur the lateral flow is accelerated from velocity $\partial h/\partial t$ on the surface of the cross-section to some velocity of approximate value $\partial h/\partial t + U\,\partial h/\partial x$ at the edge of the boundary layer. From there the velocity decreases as the radial distance increases, and tends to zero at infinity.

Even if we knew the exact distribution of the lateral velocity we could not describe this distribution by applying sufficient boundary conditions because the Oseen equations are not suitable for this purpose. The solution of the Oseen equations implies that once the lateral velocity of the cylinder is given it decreases monotonically as the radial distance increases. Hence, in order to get a good comparison with slender-body theory we assume the following boundary conditions:

$$\left.\begin{aligned} &v = 0, \quad w = V = \partial h/\partial t + U\,\partial h/\partial x, \quad \text{on} \quad r = (y^2+z^2)^{\frac{1}{2}} = a,\\ &v = 0, \quad w = 0, \quad p = p_\infty, \quad \text{at} \quad r = (y^2+z^2)^{\frac{1}{2}} = \infty,\end{aligned}\right\} \tag{9}$$

where $a$ is the radius of the cylinder. Let us also assume that $h(x, t)$ is small and has the form of a simple harmonic oscillation, i.e.

$$h(x,t) = \exp\left[i(\Omega t - \kappa x)\right] F(x), \quad \partial F/\partial x \ll 1, \tag{10}$$

where $\Omega$ is the angular velocity, $\kappa$ is the wavenumber and $F(x)$ is the envelope of the amplitudes and, in general, is a complex function.

The solution of (6) and (7) using the boundary conditions (9) and (10) was given by Lavie (1970*a*) in the following form:

$$\psi = V(x,t)\frac{\sin\theta}{K_0(ca)}\left\{\frac{2}{c}K_1(cr)-\frac{a^2}{r}K_2(ca)\right\}, \tag{11}$$

where $$V(x,t) = \frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x},$$

$$p = \rho a^2\left(\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}\right)\frac{K_2(ca)}{K_0(ca)}\frac{\cos\theta}{r}, \tag{12}$$

$$\Gamma = V(x,t)\,c\frac{K_1(cr)}{K_0(ca)}\sin\theta. \tag{13}$$

$\psi$ is the stream function defined by (6), $\Gamma$ is the vorticity function, $K_0$, $K_1$ and $K_2$ are the Bessel functions of orders zero, one and two defined in Watson (1962) and the coefficient $c$ is defined by

$$c^2 = i(\Omega-\kappa U)/\nu. \tag{14}$$

The forces $F_z$ and $F_y$ acting on the cylinder in the $z$ and in the $y$ directions are given by

$$\left.\begin{aligned} F_z &= -\int_0^{2\pi}(ap\cos\theta+2\rho\nu a\Gamma\sin\theta)\,d\theta,\\ F_y &= -\int_0^{2\pi}(ap\sin\theta-2\rho\nu a\Gamma\cos\theta)\,d\theta.\end{aligned}\right\} \tag{15}$$

Substituting the values of $p$ and $\Gamma$ in (15) and evaluating the integrals one gets

$$\left.\begin{aligned} F_y &= 0, \quad A = \pi a^2,\\ F_z &= -\rho A\left\{\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}\right)\frac{K_2(ca)}{K_0(ca)}+\frac{2\nu c}{a}V\frac{K_1(ca)}{K_0(ca)}\right\}.\end{aligned}\right\} \tag{16}$$

Note that $F_z$ is actually the lift and in the following argument we shall use the notation $L(x,t)$ for this force. The functions $K_0$, $K_1$ and $K_2$ can be expanded asymptotically, giving

$$\left.\begin{aligned} \frac{K_2(ca)}{K_0(ca)} &= 1+\frac{2}{ca}+\frac{126}{128(ca)^2}+\ldots,\\ \frac{K_1(ca)}{K_0(ca)} &= 1+\frac{1}{2ca}-\frac{9}{64(ca)^2}+\ldots.\end{aligned}\right\} \tag{17}$$

By substituting (17) into (16) and using the definition (14) we can simplify the result for the lift:

$$L(x,t) = -\rho A\,\mathrm{Re}\left\{\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}\right)\left[1+\frac{\sqrt{2}}{a}\left(\frac{\nu}{\Omega-\kappa U}\right)^{\frac{1}{2}}(1-i)-i\frac{126}{128}\frac{\nu}{a^2(\Omega-\kappa U)}+\ldots\right]\right.$$
$$\left.+\frac{\nu}{a^2}V\left[\sqrt{(2)}\,a\left(\frac{\Omega-\kappa U}{\nu}\right)(1+i)+1-\frac{9}{32a}\left(\frac{\nu}{\Omega-\kappa U}\right)^{\frac{1}{2}}\frac{1-i}{\sqrt{2}}+\ldots\right]\right\},$$

where Re means the real part. If $V(x,t)$ is real we can write $L(x,t)$ also as

$$\left.\begin{aligned} L(x,t) &= -\rho A c_1 \left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x}\right) - \rho A c_2 V, \\ c_1 = 1+\left[\frac{2\nu}{a^2(\Omega-\kappa U)}\right]^{\frac{1}{2}} &+ \ldots, \quad c_2 = \left[\frac{2\nu(\Omega-\kappa U)}{a^2}\right]^{\frac{1}{2}}. \end{aligned}\right\} \tag{18}$$

The propulsion force $P$, which causes the cylinder to swim, is calculated as follows:

$$P = \int_0^l L(x,t)\frac{\partial h}{\partial x}dx = -\rho A\int_0^l \left\{c_1\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x}\right) + c_2 V\right\}\frac{\partial h}{\partial x}dx. \tag{19}$$

The first part of (19) is calculated by integrations by parts:

$$\begin{aligned} \int_0^l \left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x}\right)\frac{\partial h}{\partial x}dx &= \int_0^l \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(V\frac{\partial h}{\partial x}\right)dx - \int_0^l \frac{\partial V}{\partial x}V\,dx \\ &= \int_0^l \frac{\partial}{\partial t}\left(V\frac{\partial h}{\partial x}\right)dx + U\left[V\frac{\partial h}{\partial x}\right]_0^l - \tfrac{1}{2}[V^2]_0^l. \end{aligned}$$

We are interested in the mean of the thrust force over a long time. Obviously, when $P$ is averaged, the first term of the above expression becomes zero. Thus we have the following expression for the average thrust $\overline{P}$:

$$\overline{P} = -\rho A c_1 U\overline{\left[V\frac{\partial h}{\partial x}\right]_0^l} + \tfrac{1}{2}\rho A c_1 [\overline{V^2}]_0^l - \rho A c_2 \int_0^l \overline{V\frac{\partial h}{\partial x}}dx, \tag{20}$$

where the bars over the terms denote their mean value over a long time.

The average power $\overline{W}$ required for swimming was found (Lavie 1970*a*) to be

$$\overline{W} = \rho U A c_1 \overline{\left[\frac{\partial h}{\partial t}\left(\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x}\right)\right]_0^l} + \rho A c_2 \overline{\int_0^l \frac{\partial h}{\partial t}V\,dx}. \tag{21}$$

The efficiency $\eta$ of the swimming is related to the following expression:

$$\eta = \overline{P}U/\overline{W}, \tag{22}$$

where $\overline{P}$ and $\overline{W}$ are given by (20) and (21).

## 3. The swimming of the Pod

As has been mentioned already, Longman & Lavie (1966) analysed the theory of the swimming of the Pod by using the hydrodynamic forces derived by Lighthill (1960*b*) from slender-body theory (see equations (4)). In order to improve their results, they suggested the inclusion in the lift force of an additional term proportional to $\mu\partial h/\partial t$, where $\mu$ is the dynamic viscosity of the liquid. The results for the hydrodynamic forces obtained in the last paragraph are much more appropriate for the swimming of the Pod than the results obtained from slender-body theory. Therefore we shall try to improve the theory of the Pod by including the local lift $L(x,t)$ (see (18)) in the equation of lateral movements (1) and also by calculating the thrust force and the required power according to (20) and (21).

In order to make equations (1) tractable Longman & Lavie made the following assumptions about the Pod (see also figure 1).

(i) $A(x) = A$ = constant over the main body of the tail. It is true that it may be desirable to have $A(l)$ different from $A(0)$, but nevertheless $A(x)$ will be kept constant over the main body of the tail. It is probably sufficient to vary $A(x)$ only over the rigid head of the permanent magnet (where (1) does not apply) in any case.

(ii) $m(x) = m$ = constant; $I(x) = I$ = constant – in agreement with assumption (i).

(iii) Except in the immediate neighbourhood of $x = 0$, the amplitude of the transverse velocity $\partial h/\partial t$ of the Pod is much higher than that of the relative fluid velocity $U\,\partial h/\partial x$ produced by the small instantaneous slope of the tail, i.e.†

$$\frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}\sim\frac{\partial h}{\partial t}. \tag{23}$$

(iv) The small magnetic head of the Pod was considered to be executing small angular vibrations about a fixed pivot located at the origin $O$. As a simulation of the moment applied by the external electromagnetic field to the magnetic head it was assumed that

$$F(x,t) = k\sin(\Omega t+\beta)\{\delta(x-\epsilon)\}/\epsilon, \tag{24}$$

where $\epsilon$ is small and tends to zero, while the hypothetical pivot was assumed to be at $x = 0$. $k$ is the amplitude of the alternating moment having the form of a standing wave with angular velocity $\Omega$ and phase angle $\beta$. Substituting (18) and (19) into (1) and using the preceding assumptions we obtain the following differential equation:

$$(m+\rho Ac_1)\frac{\partial^2 h}{\partial t^2}+\rho Ac_2\frac{\partial h}{\partial t}+EI\frac{\partial^4 h}{\partial t^4} = k\sin(\Omega t+\beta)\frac{\delta(x-\epsilon)}{\epsilon}. \tag{25}$$

The above equation was solved by Longman & Lavie (1966) but using $(m+\rho A)$ instead of $(m+\rho Ac_1)$ and, furthermore, it was assumed that $\rho Ac_2 = k$ = constant (note that, in our case, according to (31) either $c_2$ or $c_1$ is dependent on the frequency $\Omega$ and on the kinematic viscosity). Hence, following Longman & Lavie, the solution of equation (25) is

$$\begin{aligned}h(x,t) = \sum_{n=1}^{\infty}\left\{Sh\frac{\alpha_n x}{l}\Big/\sinh\alpha_n+(-1)^n\sqrt{2}\sin\frac{\alpha_n x}{l}\right\}\Big\{A_n\exp(-\tfrac{1}{2}\overline{\nu}t)\cos t\,(\omega_n^2-\tfrac{1}{4}\overline{\nu}^2)^{\frac{1}{2}}\\ +B_n\exp(-\tfrac{1}{2}\overline{\nu}t)\sin t\,(\omega_n^2-\tfrac{1}{4}\overline{\nu}^2)^{\frac{1}{2}}+\frac{Q_n(\omega_n^2-\Omega^2)}{(\omega_n^2-\Omega^2)^2+\Omega^2\overline{\nu}^2}\sin(\Omega t+\beta)\\ -\frac{Q_n\Omega\overline{\nu}}{(\omega_n^2-\Omega^2)^2+\Omega^2\overline{\nu}^2}\cos(\Omega t+\beta)\Big\},\end{aligned} \tag{26}$$

† The assumption (23) implies some restrictions. As we said before, one must distinguish between two Reynolds numbers: one related to the uniform stream velocity $U$ and the other related to the lateral velocity of the cross-section. The Oseen equations imply that $\partial h/\partial t \ll U$ so that $U\,\partial h/\partial x \ll \partial h/\partial t \ll U$. Let us suppose that at given $x$ we have $h = a\sin\omega t$, $\partial h/\partial t = a\omega\cos\omega t$. The derivative $\partial h/\partial x$ will be of the order $a/l$, where $l$ is the length of the body. In addition to that we remember that the Oseen equations are valid to about $Re = 5$ which means $Re = (d\,\partial h/\partial t)/\nu = awd/\nu = 5$, where $d$ is the diameter of the body. It follows therefore from assumption (23) together with the validity of the Oseen equations that

$$a/l \ll a\omega/U \ll 1;\quad a \leqslant 5\nu/\omega d.$$

$$Q_n = k(-1)^n \alpha_n \sqrt{2}/(m+\rho A c_1)\, l^2,$$

$$\alpha_n = (n+\tfrac{1}{4})\pi, \quad \omega_n = \alpha_n^2 a/l^2, \quad a^2 = EI/(m+\rho A c_1) \tag{27}$$

$$\bar{\nu} = \frac{\rho A c_2}{m+\rho A c_1} \approx \frac{\rho A}{m+\rho A} \frac{(2\nu\Omega)^{\frac{1}{2}}}{a+[\rho A/(m+\rho A)]\,(2\nu/\Omega)^{\frac{1}{2}}}. \tag{28}$$

Note that since the external force $F(x,t)$ has the form of a standing wave, $h(x,t)$ represents also transversal movements of standing waves. Therefore the wavenumber $\kappa$ in (18) is zero and the constants $c_1$ and $c_2$ have, in our particular case, the form

$$c_1 = 1+(2\nu/a^2\Omega)^{\frac{1}{2}}+\ldots, \quad c_2 = (2\nu\Omega/a^2)^{\frac{1}{2}}+\ldots.$$

The constants $A_n$ and $B_n$ in (26) depend on the initial conditions of the problem, but these terms are damped out by the exponentially decaying terms, and for steady operation we can drop the free oscillation and obtain

$$h(x,t) = \sum_{n=1}^{\infty} \left\{\sinh \frac{\alpha_n x}{l} \Big/ Sh\alpha_n + (-1)^n \sqrt{2} \sin \frac{\alpha_n x}{l}\right\}$$
$$\times \left\{\frac{Q_n(\omega_n^2-\Omega^2)}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2} \sin(\Omega t+\beta) - \frac{Q_n \Omega \bar{\nu}}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2} \cos(\Omega t+\beta)\right\}. \tag{29}$$

In order to calculate the thrust $\bar{P}$ and the power $\bar{W}$ we use the following relations:

$$\overline{V\frac{\partial h}{\partial x}} = U\overline{\left(\frac{\partial h}{\partial x}\right)^2}, \quad \overline{V\frac{\partial h}{\partial t}} = \overline{\left(\frac{\partial h}{\partial t}\right)^2}, \quad \overline{V^2} = \overline{\left(\frac{\partial h}{\partial t}\right)^2} + U^2 \overline{\left(\frac{\partial h}{\partial x}\right)^2}, \tag{30}$$

where the mean-square values of the above derivatives at $x = 0$ and at $x = l$ are obtained directly from (29).

$$\overline{\left(\frac{\partial h}{\partial t}\right)^2_{x=0}} = 0,$$

$$\overline{\left(\frac{\partial h}{\partial x}\right)^2_{x=0}} = \frac{2k^2}{\bar{m}^2 l^6}\left\{\left[\sum_{n=1}^{\infty} \frac{\alpha_n^2(\omega_n^2-\Omega^2)}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2}\right]^2 + \Omega^2\bar{\nu}^2\left[\sum_{n=1}^{\infty} \frac{\alpha_n^2}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2}\right]^2\right\}$$
$$= \frac{2k^2}{\bar{m}^2 l^6}\{A_{\mathrm{I}}^2 + \Omega^2\bar{\nu}^2 B_{\mathrm{I}}^2\}, \tag{31}$$

$$\overline{\left(\frac{\partial h}{\partial t}\right)^2_{x=l}} = \frac{4k^2\Omega^2}{\bar{m}^2 l^4}\left\{\left[\sum_{n=1}^{\infty} \frac{\alpha_n(-1)^n(\omega_n^2-\Omega^2)}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2}\right]^2 + \Omega^2\bar{\nu}^2\left[\sum_{n=1}^{\infty} \frac{\alpha_n(-1)^n}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2}\right]^2\right\}$$
$$= \frac{4k^2\Omega^2}{\bar{m}^2 l^4}\{C_{\mathrm{I}}^2 + \Omega^2\bar{\nu}^2 D_{\mathrm{I}}^2\}, \tag{32}$$

$$\overline{\left(\frac{\partial h}{\partial x}\right)^2_{x=l}} = \frac{4k^2}{\bar{m}^2 l^6}\left\{\left[\sum_{n=1}^{\infty} \frac{\alpha_n^2(-1)^n(\omega_n^2-\Omega^2)}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2}\right]^2 + \Omega^2\bar{\nu}^2\left[\sum_{n=1}^{\infty} \frac{\alpha_n^2(-1)^n}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2}\right]^2\right\}$$
$$= \frac{4k^2}{\bar{m}^2 l^6}\{A_{\mathrm{II}}^2 + \Omega^2\bar{\nu}^2 B_{\mathrm{II}}^2\}, \tag{33}$$

where $\bar{m} = m+\rho A c_1$. We still need the values of the integrals

$$\overline{\int_0^l \left(\frac{\partial h}{\partial x}\right)^2 dx} \quad \text{and} \quad \overline{\int_0^l \left(\frac{\partial h}{\partial t}\right)^2 dx}.$$

The value of
$$\int_0^l \left(\frac{\partial h}{\partial x}\right)^2 dx$$
can be computed numerically and this gives:

$$\left.\begin{aligned}
\overline{\int_0^l \left(\frac{\partial h}{\partial x}\right)^2 dx} &= \frac{k^2}{\overline{m}^2 l^6} \sum_{i=0}^{M-1} \{A_{\mathrm{II}m}^2 + \Omega^2 \overline{\nu}^2 B_{\mathrm{II}m}^2\},\\
A_{\mathrm{II}m} &= \sum_{n=1}^{\infty} \frac{\alpha_n^2(-1)^n(\omega_n^2-\Omega^2)\, S_{n,i}}{(\omega_n^2-\Omega^2)^2+\Omega^2\overline{\nu}^2},\\
B_{\mathrm{II}m} &= \sum_{n=1}^{\infty} \frac{\alpha_n^2(-1)^n S_{n,i}}{(\omega_n^2-\Omega^2)^2+\Omega^2\overline{\nu}^2},\\
S_{n,i} &= \frac{l}{M}\left\{\cosh \frac{\alpha_n(i+\frac{1}{2})}{M} \bigg/ \sinh \alpha_n + (-1)^n \sqrt{2} \cos \frac{\alpha_n(i+\frac{1}{2})}{M}\right\},
\end{aligned}\right\} \tag{34}$$

where $M$ is some large number which depends on the designed computation accuracy. The integral
$$\int_0^l \left(\frac{\partial h}{\partial t}\right)^2 dx$$
can be computed analytically owing to certain orthogonality conditions satisfied by (25). These conditions are explained in the article of Longman & Lavie and also in Bisplinghoff, Ashley & Halfman (1955). Thus

$$\overline{\int_0^l \left(\frac{\partial h}{\partial t}\right)^2 dx} = l\Omega^2 \sum_{n=1}^{\infty} \left\{\frac{Q_n(\omega_n^2-\Omega^2)}{(\omega_n^2-\Omega^2)^2+\Omega^2\overline{\nu}^2} \cos(\Omega t+\beta) + \frac{Q_n \Omega\overline{\nu}}{(\omega_n^2-\Omega^2)^2+\Omega^2\overline{\nu}^2}\right\}^2.$$

This brings us to
$$\overline{\int_0^l \left(\frac{\partial h}{\partial t}\right)^2 dx} = \frac{k^2\Omega^2}{\overline{m}^2 l^4}\{C_{\mathrm{II}}^2 + \Omega^2\overline{\nu}^2 D_{\mathrm{II}}^2\}, \tag{35}$$

where
$$C_{\mathrm{II}}^2 = l \sum_{n=1}^{\infty} \frac{\alpha_n^2(\omega_n^2-\Omega^2)^2}{\{(\omega_n^2-\Omega^2)^2+\Omega^2\overline{\nu}^2\}^2}, \tag{36}$$
$$D_{\mathrm{II}}^2 = l \sum_{n=1}^{\infty} \frac{\alpha_n^2}{\{(\omega_n^2-\Omega^2)^2+\Omega^2\overline{\nu}^2\}^2}.$$

With the above results we find the average thrust $\overline{P}$ and the average power $\overline{W}$ according to (19) and (21):

$$\begin{aligned}
\overline{P} = \rho A c_1 \frac{2k^2\Omega^2}{\overline{m}^2 l^4}\bigg\{C_{\mathrm{I}}^2 + \Omega^2\overline{\nu}^2 D_{\mathrm{I}}^2 - \frac{U^2}{l^2\Omega^2}[A_{\mathrm{II}}^2 + \Omega^2\overline{\nu}^2 B_{\mathrm{II}}^2 - \tfrac{1}{2}(A_{\mathrm{I}}^2 + \Omega^2\overline{\nu}^2 B_{\mathrm{I}}^2)]&\\
- U\frac{c_2}{2c_1 l^2\Omega^2} \sum_{i=0}^{M-1} (A_{\mathrm{II}m}^2 + \Omega^2\overline{\nu}^2 B_{\mathrm{II}m}^2)\bigg\},&
\end{aligned} \tag{37}$$

$$\overline{W} = \rho U A c_1 \frac{4k^2\Omega^2}{\overline{m}^2 l^4}\left\{C_{\mathrm{I}}^2 + \Omega^2\overline{\nu}^2 D_{\mathrm{I}}^2 + \frac{c_2}{4Uc_1}(C_{\mathrm{II}}^2 + \Omega^2\overline{\nu}^2 D_{\mathrm{II}}^2)\right\}. \tag{38}$$

At this point we have to make a few remarks about the thrust and the power equations (37) and (38). $\overline{W}$ is the power required to produce the transverse movement $h(x,t)$. The power $W_s$ supplied by the electromagnetic field can be calculated from
$$W_s = \int_0^l F(x,t)\frac{\partial h}{\partial t}\,dx, \tag{39}$$

where $F(x,t)$ is the external force induced by the electromagnetic field and its value is defined by (24). Thus

$$W_s = \int_0^l k\sin(\Omega t+\beta)\frac{\delta(x-\epsilon)}{\epsilon}\frac{\partial h}{\partial t}dx = \frac{k\sin(\Omega t+\beta)}{\epsilon}\frac{\partial h(\epsilon)}{\partial t}.$$

When $\epsilon \to 0$, we get

$$\begin{aligned} W_s &= \lim_{\epsilon\to 0}\frac{k\sin(\Omega t+\beta)}{\epsilon}\frac{\partial h(\epsilon)}{\partial t} \\ &= k\sin(\Omega t+\beta)\sum_{n=1}^{\infty}(-1)^n\sqrt{(2)}\frac{\alpha_n}{l}\left\{\frac{Q_n\Omega(\omega_n^2-\Omega^2)}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2}\cos(\Omega t+\beta)\right. \\ &\qquad \left.+\frac{Q_n\Omega^2\bar{\nu}}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2}\sin(\Omega t+\beta)\right\}. \end{aligned}$$

To find the average value of $W_s$ we recall that the electromagnetic field supplies energy even when the product of $\sin(\Omega t+\beta)$ and $\cos(\Omega t+\beta)$ is negative, hence

$$\overline{W}_s = \frac{k^2\Omega^2}{\overline{m}l^3}\sum_{n=1}^{\infty}\alpha_n^2\frac{(\omega_n^2-\Omega^2)+\Omega\bar{\nu}}{(\omega_n^2-\Omega^2)^2+\Omega^2\bar{\nu}^2}. \tag{40}$$

The swimming velocity can be found by comparing the thrust $\overline{P}$ (see (37)) to the drag $D$ in (3). To illustrate this, for the moment we ignore the viscosity $\bar{\nu}$ and confine ourselves to regions when $\Omega \neq \omega_n$. Then

$$\overline{P} \simeq \rho A\frac{2k^2\Omega^2}{\overline{m}^2l^4}\left\{C_{\mathrm{I}}^2-\frac{U^2}{\Omega^2l^2}[A_{\mathrm{II}}^2-\tfrac{1}{2}A_{\mathrm{I}}^2]\right\} = \tfrac{1}{2}\rho SC_D U^2,$$

or

$$U^2 = \frac{2\rho Ak^2\Omega^2C_{\mathrm{I}}^2/\overline{m}^2l^4}{\tfrac{1}{2}\rho SC_D+(2\rho Ak^2/\overline{m}^2l^6)(A_{\mathrm{II}}^2-\tfrac{1}{2}A_{\mathrm{I}}^2)}. \tag{41}$$

From (41) we see that $U$ is proportional to $\Omega$, and one would expect $U$ to increase without bound as $\Omega$ assumes large values. This however is impossible because

$$\overline{W}_s = \frac{k^2\Omega}{\overline{m}l^3}\sum_{n=1}^{\infty}\frac{\alpha_n^2}{(\omega_n^2-\Omega^2)} \geqslant \overline{W} = \rho AU\frac{4k^2\Omega^2}{\overline{m}^2l^4}C_{\mathrm{I}}^2$$

for $\nu = 0$, $\Omega \neq \omega_n$.

If we assume that (41) is correct it follows that $\overline{W}$ would increase as $\Omega^3$ while $\overline{W}_s$ would only increase like $\Omega$. Let us assume that

$$\overline{W} = \eta_1\overline{W}_s, \tag{42}$$

where $\eta_1$ is the efficiency of the conversion of electromagnetic power into mechanical power. Thus

$$U = \frac{\overline{m}l\eta_1}{4\rho A\Omega C_{\mathrm{I}}^2}\sum_{n=1}^{\infty}\frac{\alpha_n^2}{\omega_n^2-\Omega^2} \tag{43}$$

and it follows that from the power point of view the velocity $U$ would decrease as $\Omega$ increases, regarding $\eta_1$ as independent of $\Omega$. In practice we would expect $U$ to increase with $\Omega$ until $\Omega$ reaches certain value $\Omega_{cr}$; then it might decrease as $\Omega$ increases further. This behaviour of $U$ was also noticed from experimental results, Lavie (1970*b*).

## 4. Discussion of results

In §2 the hydrodynamic forces acting on a circular flexible cylinder were derived. It was assumed there that the cross-sectional area $A$ remains constant over the whole length of the cylinder. The results obtained may be generalized to the case of slender cylinders having a local cross-section $A(x)$ which varies along the longitudinal axis of the body. Since the body is slender and $A(x)$ varies slowly we note that, locally, the body shape differs little from that of an infinite cylinder whose cross-section is $A(x)$ throughout its entire length. Referring to equations (22), we can write the potential $\Phi$ for the more general case when $\rho A$ is not a constant, but represents the virtual mass $\rho \tilde{A}(x)$, as

$$\Phi = V(x,t)\,\tilde{A}(x)\frac{K_2(ca)}{\pi K_0(ca)}\frac{\cos\theta}{r}. \tag{44}$$

The pressure $p$ therefore becomes

$$p = \rho\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)(V\tilde{A}(x))\frac{K_2(ca)}{\pi K_0(ca)}\frac{\cos\theta}{r}, \tag{45}$$

where $a$ is the local radius.

Using equation (45) to calculate the local lift in (25) we obtain

$$L(x,t) = -\rho c_1\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)(V(x,t)\,\tilde{A}(x)) - \rho c_2\,V(x,t)\,\tilde{A}(x) \tag{46}$$

and consequently find that

$$\overline{P} = -\rho c_1\left\{\overline{V\tilde{A}(x)\frac{\partial h}{\partial x}}\right\}_0^l + \tfrac{1}{2}\rho c_1\overline{\{\tilde{A}(x)V^2\}_0^l} - \rho c_2\int_0^l \overline{\tilde{A}(x)\,V\frac{\partial h}{\partial x}}\,dx, \tag{47}$$

$$\overline{W} = \rho U c_1\left\{\overline{\tilde{A}(x)\frac{\partial h}{\partial t}\left(\frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}\right)}\right\}_0^l + \rho c_2\int_0^l \overline{\tilde{A}(x)\,V\frac{\partial h}{\partial t}}\,dx. \tag{48}$$

Comparing (46) and (47) with the results obtained from slender-body theory (equations (4)) we see that when the viscosity $\nu$ is zero, and thus $c_1 = 1$ and $c_2 = 0$, the results are identical. Note that the apparent mass $\rho\tilde{A}(x)$ can be applied also to the case where the body swims in liquid bounded by a circular rigid boundary, such a pipe. In this case

$$\tilde{A}(x) = \frac{b^2+a^2(x)}{b^2-a^2(x)}\pi a^2(x), \tag{49}$$

where $a(x)$ is the local radius of the swimming body and $b$ is the radius of the pipe. However, this is true when $b$ is much larger than $a(x)$. When $a(x)$ approaches $b$ a further investigation is required because the effects of viscosity.

We can now modify the results obtained for the Pod in the more general case when its cross-sectional area varies slowly such that $\tilde{A}(0) \neq \tilde{A}(l)$. We seek the equation for the swimming velocity $U$ by comparing the thrust $\overline{P}$ with the drag $D$, i.e.

$$\overline{P} = \overline{D} = \tfrac{1}{2}\rho C_D S U^2.$$

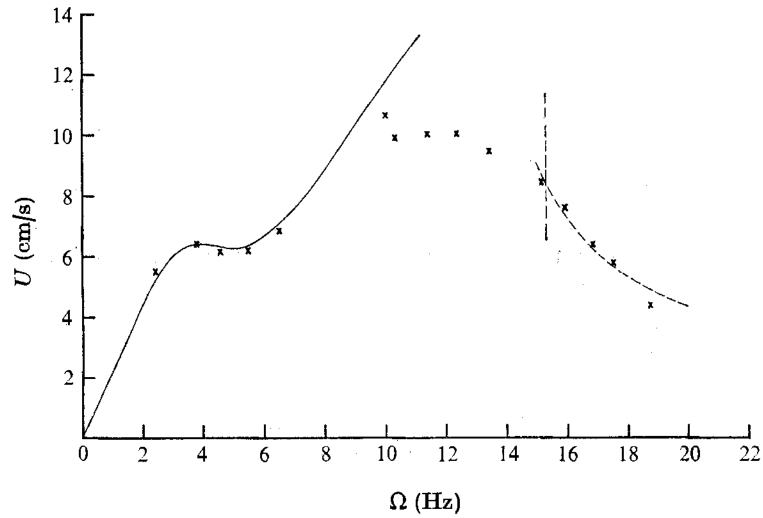


FIGURE 2. The swimming velocity of Pod $A$ as a function of the frequency.

Let the reference area $S$ be equal to $\tilde{A}(l)$, then we find the following quadratic equation for the velocity $U$:

$$U^2\left\{\tfrac{1}{2}C_D+\frac{2k^2c_1}{\overline{m}^2l^6}\left[(A_{\mathrm{II}}^2+\Omega^2\overline{\nu}^2B_{\mathrm{II}}^2)-\frac{\tilde{A}(0)}{2\tilde{A}(l)}(A_{\mathrm{I}}^2-\Omega^2\overline{\nu}^2B_{\mathrm{I}}^2)\right]\right\}$$
$$+U\frac{k^2c_2}{\overline{m}^2l^6}\sum_{i=0}^{M-1}\tilde{A}\left(\frac{i+\frac{1}{2}}{M}l\right)(A_{\mathrm{II}m}^2+\Omega^2\overline{\nu}^2B_{\mathrm{II}m}^2)$$
$$-\frac{2kc_1\Omega^2}{\overline{m}^2l^4}(C_{\mathrm{I}}^2+\Omega^2\overline{\nu}^2D_{\mathrm{I}}^2)=0. \tag{50}$$

Equation (50) already takes into account the fact that $\tilde{A}(0) \neq \tilde{A}(l)$ and assumes that the drag coefficient $C_D$ is referred to the cross-section $\tilde{A}(l)$. As we pointed out previously (50) is expected to be true while $U$ is proportional to $\Omega$ (see (41)). Afterwards it is expected that $U$ will remain constant for certain values of $\Omega$ and that further increase in $\Omega$ will cause the velocity $U$ to decrease according to (43).

At the Weizmann Institute of Science, experimental tests were conducted to measuring the swimming velocity of Pods of different sizes (Lavie 1970*b*). Typical results for the velocity $U$ as a function of $\Omega$ are given in figures 2 and 3 for two Pods. The first one was 52·7 mm long, its diameter was 3·5 mm and the permanent magnet was Alnico 5, 13·3 mm long. The second Pod was 70·8 mm long. The solid and the dotted lines show the expected behaviour of the Pod (which was calculated) and the points describe the experimental results. The resonance frequency of the first Pod was about 11 Hz. The current wave's shape in the electric coil was not sinosoidal and it happened that the third harmonic played an important role. This harmonic also explains the pick obtained at one third of the first natural frequency.

The efficiency of the swimming of the Pod can be calculated according to (36).

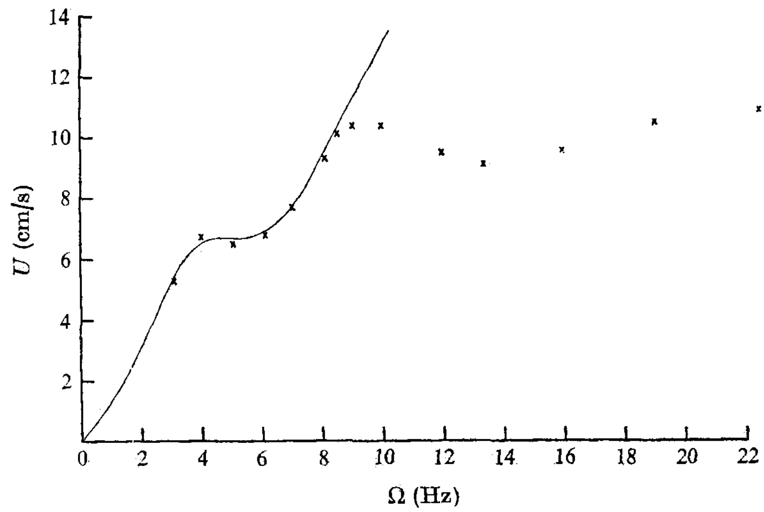


FIGURE 3. The swimming velocity of Pod $B$ as a function of the frequency.

Although the expressions for $\overline{P}$ and for $\overline{W}$ are quite complicated it can be shown that the efficiency will be less than 0·5. Lighthill (1960*a*) has pointed out that the swimming efficiency of any body performing lateral movements of standing-wave form will be at most 0·5. Therefore, in order to increase the efficiency of the Pod, a lateral movement in the form of travelling wave such as

$$h(x,t) = g(x)\cos(\omega t - kx)$$

must somehow be produced. These movements cannot be produced by the present construction of the Pod system.

Another important problem concerning the Pod is how to increase the supplied power $W_s$. This power is only a small fraction of the total power of the electromagnetic field produced by an electric coil. One way to increase $W_s$ is, of course, to increase the current in the coil and consequently to increase the amplitude $k$ of the applied moment. Another possibility is to install more than one permanent magnet in the plastic tail. This would require a special construction of the Pod, and special considerations about the interference between the magnets and the lateral movement would have to be made. Lastly it is desirable to push away the critical angular velocity and to use the mechanism of absorbing energy from the electromagnetic field more efficiently. However, for a Pod with given dimensions it seems that increasing $\Omega_{cr}$ is not practicable.

## 5. Conclusions

Taking into account both viscous and inertial forces, the hydrodynamic forces acting on deformable slender bodies placed in a uniform stream have been found. Then the results have been applied to derive the theory of the swimming of a Pod and of elastic bodies excited by an external force. Methods for increasing the swimming velocity of these bodies have been suggested.

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